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Nekrasov functions from exact Bohr–Sommerfeld periods: the case of $SU(N)$

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Abstract
We suggested in 2009 that the Nekrasov function with one non-vanishing deformation parameter ϵ is obtained by the standard Seiberg–Witten (SW) contour-integral construction. The only difference is that the SW differential $p\mathrm{d}x$ is substituted by its quantized version for the corresponding integrable system, and contour integrals become exact monodromies of the wavefunction. This provides an explicit formulation of the earlier guess by Nekrasov and Shatashvili in 2009. In this paper, we successfully check this suggestion in the first order in ϵ^2 and the first order in instanton expansion for the $SU(N)$ model, where the consistency of the so-deformed SW equations is already non-trivial.

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1. Introduction

Integrability plays a very important role in modern theoretical physics, because effective actions of quantum theories always exhibit integrability properties [3]. The basic reason for this is the freedom to change integration variables in functional integral. If this freedom is preserved on some 'mini-superspace' (moduli space) of coupling constants, the universality classes of effective actions are labeled by some simple and well-known integrable system in low spacetime dimensions. Today, there are a number of interesting examples where this phenomenon manifests itself. One of them is the Seiberg–Witten (SW) theory, describing the low-energy effective actions of 4D $\mathcal{N} = 2$ supersymmetric gauge theories [4]: universality classes in this case are labeled by 1D integrable systems [5] like Toda [5, 6], Calogero [7], Ruijsenaars [8] models and spin chains [9]. An alternative description of the SW theory is in terms of the Nekrasov functions [10], which originally appeared from an attempt to perform a regularized integration [11] over instanton moduli spaces with the help of the Duistermaat–Heckman (localization) technique [12]. Nowadays, the Nekrasov functions have become an important class of special functions in string theory [13], generalizing the ordinary hypergeometric series in a nontrivial way [14], and the AGT conjecture [15] implies that they provide a good starting point to describe at least the entire set of 2D conformal blocks. All this makes the description of the Nekrasov functions in terms of integrability theory an important and urgent problem. Of course, from the general perspective, the Nekrasov functions are fragments of KP–Toda τ-functions, closely related to discrete matrix models [16] and combinatorics of symmetric groups [17]. However, their relevance for the SW theory implies that there should be a relation to a much simpler class of 1D integrable systems. A first guess in this direction was made in a recent paper [2], where it was suggested that introducing the ϵ parameters corresponds in some way to a direct quantization of the integrability/SW relation of [5][3]. In [1], we provided an explicit description of this quantization procedure.

The SW theory [4] defines a *prepotential* $F_{\mathrm{SW}}(\vec{a})$ from the system of equations:

$$
\begin{aligned}
a_i &= \oint_{A^i} \mathrm{d}S^{(0)} = \Pi^{(0)}_{A_i},\\
\frac{\partial F_{\mathrm{SW}}(\vec{a})}{\partial a_i} &= \oint_{B_i} \mathrm{d}S^{(0)} = \Pi^{(0)}_{B_i},
\end{aligned}
\tag{1}
$$

where contour integrals are the Bohr–Sommerfeld (BS) periods of an associated 1D integrable system [5]. The claim of [1] is that Nekrasov's prepotential $\mathcal{F}(\vec{a}|\epsilon_1)$ with one ϵ-parameter switched on (in principle, there can be arbitrary many such ϵ-parameters, though Nekrasov [10] discusses just two) is defined by *the same* system (1), only the BS presymplectic differential $\mathrm{d}S^{(0)} \approx \vec{p}\mathrm{d}\vec{q}$ is substituted by its exact quantum counterpart: the one which defines the phase of exact wavefunction of the integrable system. To emphasize that the relevant moduli $\vec{a}$ are now different (deformed), we rewrite this system in the slightly different notation:

$$
\begin{aligned}
\alpha_i &= \oint_{A^i} \mathrm{d}S = \Pi_{A_i},\\
\frac{\partial \mathcal{F}(\vec{\alpha}|\epsilon)}{\partial \alpha_i} &= \oint_{B_i} \mathrm{d}S = \Pi_{B_i}.
\end{aligned}
\tag{2}
$$

The deformed BS periods are nothing but (Abelian) monodromies of the wavefunction.

In [1], we explicitly checked this suggestion (in the lowest orders of various expansions) only in the simplest $SU(2)$ case, when the relevant integrable system is the ordinary sine-Gordon. Though generalizations to $SU(N)$ Toda systems are well known to be straightforward, this is an important check to be done, because for $N > 2$ system (2) could be non-resolvable: no set of periods can be represented as a gradient of something. Consistency of the system cannot be proved with the help of ordinary Riemann's theorem $T_{ij} = T_{ji}$ as in the case of the original SW theory, because, after the deformation, $\mathrm{d}S$ is no longer a SW differential with the property $\delta(\mathrm{d}S) =$ holomorphic. Still, a memory of the spectral Riemann surface survives (it actually becomes modified only in the vicinity of ramification points), and we gave a technical argument at the end of [1] in favor of the consistency of (2), and now we check that this system is indeed consistent and, moreover, has $\mathcal{F}(a|\epsilon_1)$ as its solution. Like [1], we will make this check only in the first orders of expansions in ϵ_1^2 and Λ^{2N}, and even this calculation is rather cumbersome. A better proof should, of course, be sought.

[3] A similar relation of 2D supersymmetric theories and quantum integrable systems can be found in [18].

To simplify the calculations, we exploit the existing knowledge about the SW theory and the Nekrasov functions as much as possible. Actually, we proceed in the following three steps.

Step 1. SW periods $\Pi^{(0)}$ and Nekrasov functions. The $SU(N)$ universality class of SW theory is labeled by a polynomial

$$K(p) = \sum_{k=0}^{N} u_k p^k = u_N \prod_{i=1}^{N} (p - \lambda_i). \tag{3}$$

The SW/Toda spectral curve is given then by

$$K(p) + \gamma \cos\phi = 0, \qquad \gamma = \Lambda^N, \tag{4}$$

and the SW differential is

$$\mathrm{d}S^{(0)} = p\,\mathrm{d}\phi. \tag{5}$$

The periods $\Pi^{(0)}$ can be calculated in various ways, either directly or with the help of the Picard–Fucks equations. We, however, take the most economic and transparent way: we calculate $a_i(\vec{\lambda})$ directly from the definition, but take the difficult dual periods from the Nekrasov function

$$F(\vec{a}) = \lim_{\epsilon_1,\epsilon_2\to 0} \epsilon_1\epsilon_2 \log Z_{\mathrm{LNS}}(\vec{a}|\epsilon_1,\epsilon_2). \tag{6}$$

Step 2. WKB theory and deformed differential dS. The deformed differential dS is an exact solution to the deformed (quantized) equation (4); see the very last formula in [1]:

$$\left\{K\left(-\mathrm{i}\hbar\frac{\partial}{\partial\phi}\right) + \gamma\cos\phi\right\}\exp\left(\frac{\mathrm{i}}{\hbar}\int^{\phi}\mathrm{d}S\right) = 0, \tag{7}$$

and actually $\hbar = \epsilon_1$. WKB theory [19] provides an expansion of $\mathrm{d}S = \sum_{k=0}^{\infty}\hbar^k P_k\,\mathrm{d}\phi$, where $P_0 = p(\phi)$ is the 'classical' momentum, that is, the root of (4), which is single valued on the spectral Riemann surface. A technically reasonable way to calculate the periods of $P_k\,\mathrm{d}\phi$ with $k > 0$ is to represent $\mathrm{d}S = \hat{\mathcal{O}}\,\mathrm{d}S^{(0)}$ as an action of some differential operator $\mathcal{O}$ (acting on parameters u_i and γ): then $\Pi_C = \hat{\mathcal{O}}\,\Pi_C^{(0)}$.

Step 3. The check of the 'exact BS' suggestion of [1]. Finally one

- evaluates the deformed A-periods $\vec{\alpha}(\vec{\lambda}) = \hat{\mathcal{O}}[\vec{a}(\vec{\lambda})]$,
- substitutes these deformed A-periods into the α-derivatives of the known Nekrasov function $\mathcal{F}(\vec{\alpha}|\epsilon_1) = \lim_{\epsilon_2\to 0}\epsilon_1\epsilon_2\log Z_{\mathrm{LNS}}(\vec{\alpha}|\epsilon_1,\epsilon_2)$,
- compares the result with deformed B-periods, obtained at step (1) from the a-derivatives of the SW–Nekrasov function $F_{\mathrm{SW}}(\vec{a}) = \mathcal{F}(\vec{a}|\epsilon_1 = 0)$, i.e., with $\hat{\mathcal{O}}[\partial_{\vec{a}}F_{\mathrm{SW}}(\vec{a})]$.

In other words, we will prove the relation

$$\boxed{\Pi_B\big(\hat{\mathcal{O}}\big[\Pi_A^{(0)}(\lambda)\big]\big) = \hat{\mathcal{O}}\big[\Pi_B^{(0)}\big(\Pi_A^{(0)}(\lambda)\big)\big]} \tag{8}$$

extracting $\Pi_B^{(0)}(a)$ and $\Pi_B(a)$ from the Nekrasov functions with vanishing and non-vanishing ϵ_1, respectively, explicitly evaluating $\Pi_A^{(0)}(\lambda)$ and deriving operator $\hat{\mathcal{O}}$ from WKB theory.

All these steps are actually easily computerized and higher-order corrections can also be analyzed after that. In this paper, however, we present as many formulae as possible explicitly, without recourse to computer calculations. In fact, there is a close similarity between emerging formulae and those familiar from various matrix-model calculations, especially from [20] and the theory of CIV–DV potentials [21].

We actually begin in section 2 from step 2, then proceed to step 1 in sections 3 and 4 and end with step 3 in section 5.

2. WKB theory and deformed differential dS

2.1. Conjugation of the differential operator

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int^x P\mathrm{d}x}(-\mathrm{i}\hbar\partial)^n\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int^x P\mathrm{d}x} = P^n - \mathrm{i}\hbar\frac{n(n-1)}{2}P^{n-2}\dot P - \hbar^2\left(\frac{n(n-1)(n-2)}{6}P^{n-3}\ddot P + \frac{n(n-1)(n-2)(n-3)}{8}P^{n-4}\dot P^2\right) + O(\hbar^3), \tag{9}$$

where $\dot P \equiv \partial P$, while prime is reserved for P-derivatives of P-dependent functions; see below.

2.2. Schrödinger equation (7) for the differential dS

For

$$K(z) = \sum_{k=0}^N u_k z^k \tag{10}$$

one needs to solve

$$(K(-\mathrm{i}\hbar\partial) + \gamma\cos x)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int^x P\mathrm{d}x} = 0. \tag{11}$$

Making use of (9), this can be rewritten as

$$K(P) - \frac{\mathrm{i}\hbar}{2}K''(P)\dot P - \hbar^2\left(\frac{1}{6}K'''(P)\ddot P + \frac{1}{8}K''''(P)\dot P^2\right) = -V(x) = -\gamma\cos x. \tag{12}$$

Substituting

$$P = p + \hbar P_1 + \hbar^2 P_2 + O(\hbar^3), \tag{13}$$

one obtains

$$\begin{aligned} &K(p) = -V(x),\\ &P_1 = -\mathrm{i}\frac{K''(p)\dot p}{2K'(p)} = -\frac{\mathrm{i}}{2}\partial(\log K'(p)),\\ &P_2 = \left(\frac{3K''^3}{8K'^3} - \frac{K''K'''}{2K'^2} + \frac{K''''}{8K'}\right)\dot p^2 + \left(-\frac{K''}{4K'^2} + \frac{K'''}{6K'}\right)\ddot p,\\ &\ldots. \end{aligned} \tag{14}$$

Here and below K with omitted argument denotes $K(p)$, similarly $K' = K'(p)$ and so on.

From the first equation it follows that

$$\begin{aligned} &\dot p = -\frac{V'}{K'},\\ &\ddot p = -\frac{V''}{K'} - \frac{K''V'^2}{K'^3},\\ &\ldots \end{aligned} \tag{15}$$

and

$$P_2 = \left(\frac{K''}{4K'^3} - \frac{K'''}{6K'^2}\right)V'' + \left(\frac{5K''^3}{8K'^5} - \frac{2K''K'''}{3K'^4} + \frac{K''''}{8K'^3}\right)V'^2. \tag{16}$$

2.3. Simplified expression for contour integrals

For contour integrals integration by parts is allowed, and this allows one to considerably simplify the integral of (16):

$$\Pi_C^{(2)} \equiv \hbar^2 \oint_C P_2\, \mathrm{d}x = \frac{\hbar^2}{24} \oint_C \left(\frac{K''^2}{K'^3} - \frac{K'''}{K'^2} \right) V''\, \mathrm{d}x. \tag{17}$$

For $K(p) = \frac{1}{2}p^2 - E$, these formulae reproduce the standard WKB expressions used in [1].

2.4. Exact periods from BS periods and the operator $\hat{\mathcal{O}}$

For $V(x) = \gamma \cos x$, one has $V'' = -V$. Further, from $K(p) = -V = -\gamma \cos x$ and (10) it follows that

$$\begin{aligned} &\gamma \frac{\partial p}{\partial \gamma} = -\frac{V}{K'}, \\ &\frac{\partial p}{\partial u_j} = -\frac{p^j}{K'}, \\ &\gamma \frac{\partial^2 p}{\partial \gamma \partial u_j} = -\left(\frac{K''}{K'^3} p^j - \frac{j p^{j-1}}{K'^2} \right) V \end{aligned} \tag{18}$$

and

$$\frac{\hbar^2 \gamma}{24} \frac{\partial}{\partial \gamma} \left(\sum_j j(j-1) u_j \frac{\partial}{\partial u_{j-2}} \right) p = -\frac{\hbar^2}{24} \left(\frac{K''^2}{K'^3} - \frac{K'''}{K'^2} \right) V. \tag{19}$$

This means that for any closed contour C

$$\boxed{\Pi_C^{(0)} + \Pi_C^{(2)} = \hat{\mathcal{O}} \Pi_C^{(0)} = \left(1 + \frac{\hbar^2 \gamma}{24} \frac{\partial}{\partial \gamma} \sum_j j(j-1) u_j \frac{\partial}{\partial u_{j-2}} \right) \Pi_C^{(0)}}\,. \tag{20}$$

3. Nekrasov functions

The Nekrasov functions are now reviewed in numerous papers [22]. They are obtained from the LNS contour multi-integrals [11], which in the simplest $SU(N)$ case look like

$$\begin{aligned} Z_{\mathrm{LNS}}(\vec{a}|\{\epsilon\}) \equiv & \sum_k \frac{1}{k!} \left(\frac{\epsilon}{\epsilon_+ \epsilon_-} \right)^k \prod_{I=1}^k \oint \frac{\mathrm{d}\varphi_I}{2\pi \mathrm{i}} \\ & \times \frac{Q(\varphi_I)}{\prod_{j=1}^N (\varphi_I - a_j)(\varphi_I - a_j + \epsilon)} \prod_{I<J}^N \frac{\varphi_{IJ}^2 \prod_{a<b} \left(\varphi_{IJ}^2 - (\epsilon_a + \epsilon_b)^2 \right) \ldots}{\prod_a \left(\varphi_{IJ}^2 - \epsilon_a^2 \right) \ldots}, \end{aligned} \tag{21}$$

where the polynomial Q depends on the matter content of the model, for pure gauge theory $Q(\varphi) = \Lambda^{2N}$. The crucial step was done in [10]: the integral was rewritten as an explicit sum over a collection of Young diagrams, which provided a practically useful expansion basis for various purposes.

The Nekrasov function for $SU(N)$ is given by

$$\mathcal{F}(a|\epsilon_1) = \mathcal{F}^{\mathrm{pert}}(a|\epsilon_1) + \mathcal{F}^{\mathrm{inst}}(a|\epsilon_1), \tag{22}$$

where the perturbative contribution for $\epsilon \neq 0$ looks nice only when the a-derivative is taken:

$$-\frac{\partial \mathcal{F}^{\text{pert}}}{\partial a_i} = 2\epsilon_1 \sum_{j\neq i} \log \frac{\Gamma(1+a_{ij}/\epsilon_1)}{\Gamma(1-a_{ij}/\epsilon_1)}$$

$$= \sum_{j\neq i} 4a_{ij} \left\{ \left(\log \frac{a_{ij}}{\Lambda} - 1\right) + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)} \left(\frac{\epsilon_1}{a_{ij}}\right)^{2m} \right\}$$

$$= 4 \sum_{j\neq i} \left\{ a_{ij} \left(\log \frac{a_{ij}}{\Lambda} - 1\right) + \frac{\epsilon_1^2}{12 a_{ij}} + O\left(\epsilon_1^4\right) \right\}, \tag{23}$$

while the instanton part is a series in powers of $\gamma^2 = \Lambda^{2N}$, of which we will need only the first term (associated with the single-box Young diagrams)

$$\mathcal{F}^{\text{inst}} = \frac{\Lambda^{2N}}{2u_N^2} \sum_{i=1}^{N} \prod_{j\neq i} \frac{1}{a_{ij}(a_{ij}+\epsilon)} + O(\Lambda^{4N})$$

$$= \frac{1}{2u_N^2} \sum_{i=1}^{N} \frac{\Lambda^{2N}}{\prod_{j\neq i} a_{ij}^2} \left\{ 1 + \epsilon^2 \left(\sum_{j\neq i} \frac{1}{a_{ij}^2} + \sum_{j<k} \frac{1}{a_{ij}a_{ik}} \right) + O(\epsilon^4) \right\} + O(\Lambda^{4N}). \tag{24}$$

The SW prepotential $F_{\text{SW}}(\vec{a})$ is defined by the same formulae, only all terms with ϵ^2 are omitted; see section 4.2.

4. SW/BS periods $\Pi^{(0)}$

As explained in the introduction, we evaluate the A-periods $a_i = \Pi^{(0)}(A^i)$ as functions of λ_i and γ directly, while the B-periods $\Pi^{(0)}(B_i)$ will be obtained from (1) by differentiating $F_{\text{SW}}(a_i)$ from the previous section and then substituting there $a_i(\vec{\lambda})$.

4.1. SW/BS A-periods $\vec{a}$ through the roots $\vec{\lambda}$

Shifting $\phi \to \phi - \mathrm{i}N \log \Lambda$ in (4), one obtains

$$\mathrm{e}^{\mathrm{i}\phi} = -(2K(p) + \Lambda^{2N}\,\mathrm{e}^{-\mathrm{i}\phi}) = -2K(p)\left(1 - \frac{\Lambda^{2N}}{4K(p)^2}\right). \tag{25}$$

Therefore,

$$\Pi^{(0)} = \mathrm{i} \oint p\,\mathrm{d}\phi = \oint \frac{p\,\mathrm{d}K}{K} + \frac{\Lambda^{2N}}{2} \oint \frac{p\,\mathrm{d}K}{K^3} = \sum_k \oint \frac{p\,\mathrm{d}p}{p-\lambda_k} + \frac{\Lambda^{2N}}{4u_N^2} \oint \frac{\mathrm{d}p}{\prod_k (p-\lambda_k)^2} \tag{26}$$

and

$$\boxed{a_i = \Pi^{(0)}_{A_i} = \lambda_i - \frac{\Lambda^{2N}}{2u_N^2 \prod_{k\neq i} \lambda_{ik}^2} \sum_{k\neq i} \frac{1}{\lambda_{ik}}}. \tag{27}$$

4.2. SW/BS B-periods from the Nekrasov function

Putting $\epsilon = 0$ in formulae of section 3, one obtains

$$\Pi^{(0)}_{B_i} = -\frac{1}{4}\frac{\partial \mathcal{F}_{\text{SW}}}{\partial a_i} = \sum_{j\neq i} a_{ij} \left(\log \frac{a_{ij}}{\Lambda} - 1\right) - \frac{\Lambda^{2N}}{8u_N^2} \frac{\partial}{\partial a_i} \sum_{j=1}^{N} \frac{1}{\prod_{k\neq j} a_{jk}^2} + O(\Lambda^{4N})$$

$$= \sum_{j\neq i} a_{ij} \left(\log \frac{a_{ij}}{\Lambda} - 1\right) + \frac{\Lambda^{2N}}{4u_N^2} \left(\frac{1}{\prod_{k\neq i} a_{ik}^2} \sum_{k\neq i} \frac{1}{a_{ik}} + \sum_{j\neq i} \frac{1}{a_{ij}^3 \prod_{k\neq i,j} a_{jk}^2} \right). \tag{28}$$

4.3. BS B-periods through the roots $\vec\lambda$

In order to apply operator $\hat{\mathcal{O}}$, one needs the periods expressed through the roots $\vec\lambda$ or coefficients $\vec u$ rather than through the moduli $\vec a$. Thus, one needs to substitute $\vec a(\vec\lambda)$ from (27) into (28)

$$\boxed{\Pi^{(0)}_{B_i} = \sum_{j\neq i} a_{ij}(\lambda)\left(\log\frac{\lambda_{ij}}{\Lambda}-1\right)+\frac{\Lambda^{2N}}{4u_N^2}\left(\frac{1}{\prod_{k\neq i}\lambda_{ik}^2}\sum_{k\neq i}\frac{1}{\lambda_{ik}}+\sum_{j\neq i}\frac{1}{\lambda_{ij}^3\prod_{k\neq i,j}\lambda_{jk}^2}\right)}. \quad (29)$$

In the one-instanton approximation, the only difference between (29) and (28), except for a simple substitution $a_i \to \lambda_i$, is that the coefficient in front of logarithm is now a_{ij}, not λ_{ij}. The change of logarithm's argument does not contribute.

5. Quantized SW prepotential and the Nekrasov function

We are now ready to act with operator (20),

$$\hat{\mathcal{O}} = \left(1+\frac{\hbar^2\gamma}{24}\frac{\partial}{\partial\gamma}\sum_j j(j-1)u_j\frac{\partial}{\partial u_{j-2}}+O(\hbar^4)\right) = 1+\frac{\epsilon_1^2}{24}\hat{\mathcal{O}}^{(2)}+O(\epsilon^4), \quad (30)$$

on (27) and (29), substitute the former one into the full Nekrasov functions (22)–(24) and compare its derivative with the latter one. The results coincide, thus validating the suggestion of [1] in the first order in Λ^{2N} and ϵ^2.

5.1. Specifics of the second-order approximation

Operator $\hat{\mathcal{O}}^{(2)}$ acts only on the Λ-dependent ($\gamma = \Lambda^N$) quantities, and the u-differential operator can be conveniently expressed through the λ-derivatives:

$$\sum_{j=0}^{N} j(j-1)u_j\frac{\partial}{\partial u_{j-2}} = -\sum_{m=1}^{N}\frac{K''(\lambda_m)}{K'(\lambda_m)}\frac{\partial}{\partial\lambda_m}. \quad (31)$$

It can easily be tested by acting on $p(u_i)$ and using $K'(p)\frac{\partial p}{\partial u_j} = -p^j$.

Identity (8), which we want to prove, in the leading approximation can be rewritten as follows. Its left-hand side is

$$\Pi_{B_i}\left(\vec a+\frac{\epsilon_1^2}{24}\hat{\mathcal{O}}^{(2)}[\vec a]\right) = \Pi^{(0)}_{B_i}(\vec a)+\frac{\epsilon_1^2}{24}\sum_{j=1}^{N}\hat{\mathcal{O}}^{(2)}[a_j(\vec\lambda)]\frac{\partial}{\partial a_j}\Pi^{(0)}_{B_i}(\vec a)$$
$$+\frac{\epsilon^2}{24}\left(2\sum_{j\neq i}\frac{1}{a_{ij}}+\frac{12\Lambda^{2N}}{u_N^2\prod_{j\neq i}a_{ij}^2}\left(\sum_{j\neq i}\frac{1}{a_{ij}^2}+\sum_{j<k}\frac{1}{a_{ij}a_{ik}}\right)\right), \quad (32)$$

while its right-hand side is

$$\Pi^{(0)}_{B_i}(\vec a)+\frac{\epsilon_1^2}{24}\hat{\mathcal{O}}^{(2)}\left[\Pi^{(0)}_{B_i}(\vec a(\vec\lambda))\right]. \quad (33)$$

Thus what we prove in this paper is

$$\hat{\mathcal{O}}^{(2)}\left[\Pi^{(0)}_{B_i}(\vec a(\vec\lambda))\right]-\sum_{j=1}^{N}\hat{\mathcal{O}}^{(2)}[a_j(\vec\lambda)]\frac{\partial}{\partial a_j}\Pi^{(0)}_{B_i}(\vec a)$$
$$= 2\sum_{j\neq i}\frac{1}{a_{ij}}+\frac{12\Lambda^{2N}}{u_N^2\prod_{j\neq i}a_{ij}^2}\left(\sum_{j\neq i}\frac{1}{a_{ij}^2}+\sum_{j<k}\frac{1}{a_{ij}a_{ik}}\right). \quad (34)$$

In the next subsection, we explicitly describe the check for Λ-independent terms in this formula. The single-instanton contributions, i.e., the terms with Λ^{2N}, also match at both sides, but formulae are somewhat lengthy and we do not present them in this paper.

5.2. *Perturbative level*

For the perturbative part of the Nekrasov function, the difference between $\vec{a}$ and $\vec{\lambda}$ is inessential. The $\hbar$-corrections ($\hbar = \epsilon_1$) to the Λ-independent piece in $\mathcal{F}(\vec{a}|\epsilon_1)$ arise from the action of deformation operator $\hat{\mathcal{O}}$ on the logarithm in perturbative part of the SW prepotential:

$$-\hat{\mathcal{O}}\frac{\partial\mathcal{F}}{\partial a_i} = \left(1+\frac{\hbar^2}{24}\gamma\frac{\partial}{\partial\gamma}\sum_k k(k-1)u_k\frac{\partial}{\partial u_{k-2}}+\cdots\right)\sum_{j\neq i}4a_{ij}\log\frac{a_{ij}}{\Lambda} \tag{35}$$

$$= 4\sum_{j\neq i}\left\{\lambda_{ij}\log\frac{\lambda_{ij}}{\Lambda}+\frac{\hbar^2}{24N}\left(\frac{K''(\lambda_i)}{K'(\lambda_i)}-\frac{K''(\lambda_j)}{K'(\lambda_j)}\right)+O(\hbar^4,\Lambda^2)\right\}. \tag{36}$$

In the last line and in the remaining part of the calculation, we neglect all the dependences on $\gamma = \Lambda^N$; in this approximation, a_i are just the roots λ_i of the polynomial $K(p) = u_N\prod_{i=1}^N(p-\lambda_i)$ and

$$\begin{aligned} K'(\lambda_i) &= u_N\prod_{j\neq i}\lambda_{ij},\\ K''(\lambda_i) &= 2u_N\sum_{j\neq i}\left(\prod_{k\neq i,j}\lambda_{ik}\right)\end{aligned} \tag{37}$$

and

$$\frac{K''(\lambda_i)}{K'(\lambda_i)} = 2\sum_{k\neq i}\frac{1}{\lambda_{ik}}. \tag{38}$$

Using these formulae, one can check that (36) coincides with (23), provided $\hbar = \epsilon_1$:

$$\boxed{\sum_{j\neq i}\left(\frac{K''(\lambda_i)}{K'(\lambda_i)}-\frac{K''(\lambda_j)}{K'(\lambda_j)}\right) = 2N\sum_{j\neq i}\frac{1}{\lambda_{ij}}}. \tag{39}$$

Indeed,

$$\begin{aligned} N=2: &\quad \frac{2}{\lambda_{12}}-\frac{2}{\lambda_{21}} = \frac{4}{\lambda_{12}},\\ N=3: &\quad 2\cdot\frac{2(\lambda_{12}+\lambda_{13})}{\lambda_{12}\lambda_{13}}-\frac{2(\lambda_{21}+\lambda_{23})}{\lambda_{21}\lambda_{23}}-\frac{2(\lambda_{31}+\lambda_{32})}{\lambda_{31}\lambda_{32}} = 6\left(\frac{1}{\lambda_{12}}+\frac{1}{\lambda_{13}}\right),\\ &\ldots.\end{aligned} \tag{40}$$

6. Conclusion

In this paper, we reported the first check of the claim that the (degenerated) Nekrasov functions are neatly described by the deformation of the SW construction from quasiclassical to quantum integrable systems in the simplest non-Abelian case of the $SU(N)$ gauge theory or the $SL(N)$ affine Toda system. Switching from the quasiclassical BS periods to the exact quantum monodromies preserves consistency of the SW system of equations; thus, they can be used to define the deformed prepotential which coincides with Nekrasov's $\mathcal{F}(\vec{a}|\epsilon_1)$ with $\epsilon_2 = 0$.

This seems to be in accordance with the original guess in [2]. We performed the check only in the first order, both in instanton corrections (in $\gamma^2 = \Lambda^{2N}$) and in the quantum deformation parameter $\hbar^2 = \epsilon_1^2$; this case is already non-trivial. Of course, higher-order corrections deserve to be found as well. Generalizations to other models with other gauge groups and additional matter multiplets, especially to quiver theories, should also be examined. Of interest is also the similar study of the second deformation to $\epsilon_1, \epsilon_2 \neq 0$ and its relation to another important hypothesis: the AGT conjecture [15].

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